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RESEARCH ARTICLE

Regular type distributions in mechanism design and ρ-concavity

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Abstract Some of the best-known results in mechanism design depend critically on Myerson's (Math Oper Res 6:58–73, 1981) regularity condition. For example, the second-price auction with reserve price is revenue maximizing only if the type distribution is regular. This paper offers two main findings. First, a new interpretation of regularity is developed—similar to that of a monotone hazard rate—in terms of being the next to fail. Second, using expanded concepts of concavity, a tight sufficient condition is obtained for a density to define a regular distribution. New examples of regular distributions are identified. Applications are discussed.

Keywords Virtual valuation · Regularity · Generalized concavity · Prékopa–Borell theorem · Mechanism design

JEL Classification D82 · D44 · D86 · C16

1 Introduction

Some of the most celebrated results in the theory of mechanism design require the underlying type distribution to be *regular*. For example, the second-price auction with reserve price is revenue maximizing only under the condition of regularity (Myerson

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1981). When the type distribution is not regular, the optimal mechanism will typically entail conditional minimum bids (Maskin and Riley 1989). In general, irregular type distributions necessitate the characterization of optimal bunches (Nöldeke and Samuelson 2007). Formally, regularity says that the *virtual valuation*,

$$J_f(x) = x - \frac{1 - F(x)}{f(x)}, \tag{1}$$

is strictly increasing in the type x, where f and F, respectively, denote the density and distribution function of the type distribution.[1]

A common way to ensure regularity is to impose that the reciprocal of the second term in (1), i.e., the *hazard rate* of the type distribution,

$$\lambda_f(x) = \frac{f(x)}{1 - F(x)}, \tag{2}$$

is monotone increasing.[2] This approach has been found useful mainly for two reasons. First, the hazard rate allows an immediate interpretation as a conditional likelihood of failure. Indeed, if $F(x)$ is the probability that a machine will fail before time x, then the hazard rate is the instantaneous probability of failure, given that the machine has not failed before time x (Barlow and Proschan 1975). Second, distributions with log-concave densities are known to possess a monotone hazard rate (An 1998). This result can be used to identify many parameterized examples of regular type distributions. Specifically, as Bagnoli and Bergström (2005) show, regularity holds for the uniform, normal, exponential, logistic, extreme-value, Laplace, Maxwell, and Rayleigh distributions. With restrictions to parameters, this list extends to power, Weibull, Gamma, Chi-squared, Chi, and beta distributions.

However, the hazard rate condition implies that virtual valuations increase with slope ≥ 1, which is overly restrictive. For example, as illustrated in Fig. 1, the log-normal distribution does not possess a monotone hazard rate, but will still be regular unless the density is very flat.[3] However, specifications with precisely this shape have been found plausible as an empirical description of bidder valuations (Baldwin et al. 1997; Guerre et al. 2000; Laffont et al. 1995). Thus, imposing the hazard rate condition or even a log-concave density not only impairs the power of theoretical findings but also restricts in a substantial way the set of distributional specifications available for applied work.

This paper offers two main findings. The first is a statistical interpretation of regularity. As in the case of the hazard rate, the density function measures the instantaneous unconditional rate of failure. Regularity can then be captured in terms of the probability that a given machine will be *the next* to fail. This yields some intuition, e.g.,

[1] Equivalently, the marginal revenue of a monopolist facing inverse demand $p = F^{-1}(1-q)$ is strictly declining in output (Bulow and Roberts 1989).

[2] For a helpful discussion of the respective classes of distributions with increasing and decreasing hazard rate, see Hoppe et al. (2011).

[3] Indeed, the log-normal distribution is regular provided its skewness is smaller than $(e^2+2)\sqrt{e^2-1} \approx 23.73$ (see Table 1 and the Appendix).

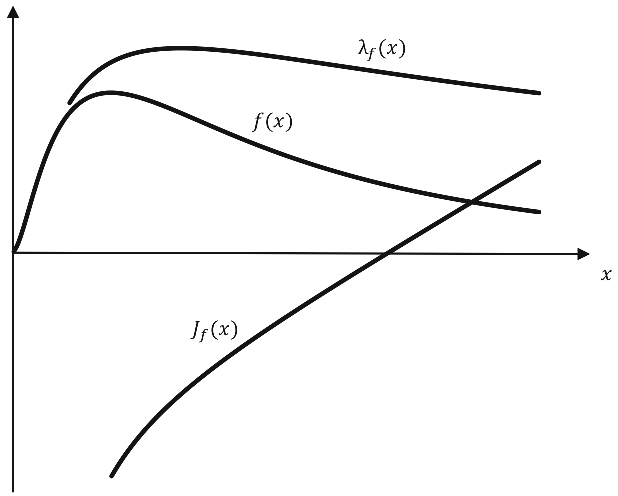

Fig. 1 Regularity of the log-normal distribution; the figure shows density, hazard rate, and virtual valuation of a random variable whose logarithm follows a standard normal distribution

regarding truncations of regular distributions. The second main result, which is the central point of the paper, is a sufficient condition for a density to define a regular distribution. The condition, referred to as strong $(-\frac{1}{2})$-concavity, is much tighter than log-concavity. Numerous new examples of distributions can be shown to be regular. In particular, we establish the regularity of distributions of log-normal shape, for which existing criteria have no bite.

The rest of the paper is organized as follows. Section 2 reviews mathematical prerequisites. An interpretation of regularity is developed in Sect. 3. In Sect. 4, we prove a general characterization of distributions that possess weakly increasing virtual valuations. Section 5 contains the key result of the paper, viz. that a strongly $(-\frac{1}{2})$-concave density defines a regular distribution. Applications are outlined in Sect. 6. Sect. 7 concludes. An Appendix provides background information on Tables 1 and 2.

2 Mathematical tools

This section reviews some mathematical concepts and results that will be used in the analysis.

2.1 Generalized concavity

A function $g \geq 0$ on $\mathbb{R}^N$ is called *ρ-concave*, for $\rho \neq 0$, if the set $X_g = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : g(x_1, \ldots, x_n) > 0\}$ is convex, and $(g(x_1, \ldots, x_N))^\rho/\rho$ is concave on X_g. For $\rho = 0$, the definition is extended by the requirement that g must be log-concave on X_g.[4]

[4] Complemented by the two limit cases $\rho = \pm\infty$ (which are not needed here), this is the definition used in the economics literature since Caplin and Nalebuff (1991a,b). Dierker (1991) is an early application of generalized concavity in the economics literature.

Higher values of ρ correspond to more stringent variants of concavity. For example, log-concavity is more stringent than ρ-concavity for any $\rho < 0$. We call a function g *strongly ρ-concave* if g is ρ'-concave for some $\rho' > \rho$. For a twice differentiable function $g : \mathbb{R} \to \mathbb{R}_{++}$ in a single variable x, the condition of ρ-concavity is equivalent to $g(x)g''(x) - (1-\rho)g'(x)^2 \leq 0$.

Among alternative notions of concavity, the definition above is highlighted by the fact that concavity properties are passed on from a density to the corresponding distribution.

Theorem 2.1 (Prékopa–Borell) *Let $g = g(x_1, \ldots, x_N) \geq 0$ be a density on $\mathbb{R}^N$. If g is ρ-concave for some $\rho > -\frac{1}{N}$, then*

$$G(z) = \int\limits_{\{x \in \mathbb{R}^N : x_N \leq z\}} g(x_1, \ldots, x_N)\, dx_1 \ldots dx_N \tag{3}$$

is $\widehat{\rho}$-concave with $\widehat{\rho} = \frac{\rho}{1+\rho N}$.

For a helpful discussion of this result, see Caplin and Nalebuff (1991a). In the simplest case ($N = 1$), the Prékopa–Borell theorem says that if a density $g \geq 0$ on $\mathbb{R}$ is ρ-concave for some $\rho > -1$, then $G(z) = \int_{-\infty}^{z} g(x)\, \mathrm{d}x$ is $\widehat{\rho}$-concave with $\widehat{\rho} = \frac{\rho}{1+\rho}$.

Theorem 2.1 is best known in the special case where $\rho = 0$ (Prékopa 1973). For example, if g is a log-concave density, then both G and $1 - G$ are log-concave, and hence, $\frac{g}{G}$ is monotone decreasing, and $\frac{g}{1-G}$ monotone increasing (An 1998; Bagnoli and Bergström 2005). Similarly, if $g(x)$ is (strictly) log-concave in $\log x$, then $\frac{xg(x)}{1-G(x)}$ is (strictly) increasing in x (van den Berg 2007; Zeng 2011). A multidimensional variant of Prékopa's theorem has been used, e.g., by Ivanov (2011).

2.2 Minimal conditions for monotonicity

A smooth function is monotone increasing provided that its first derivative is never negative. Here is a generalization to the non-differentiable case. For a given function g, denote by $\overline{g}^+(x) = \limsup_{\varepsilon \to 0+} \frac{1}{\varepsilon}(g(x+\varepsilon) - g(x))$ the right-hand upper Dini derivative at x.

Theorem 2.2 *Assume that*

$$\limsup_{\varepsilon \to 0+} g(x - \varepsilon) \leq g(x) \leq \limsup_{\varepsilon \to 0+} g(x + \varepsilon) \tag{4}$$

at any x, that $\overline{g}^+(x) \geq 0$ a.e., and that $\overline{g}^+(x) > -\infty$ except possibly at a countable set. Then g is monotone increasing.

This result follows from Theorem 7.3 in Saks (1937).[5] Note that (4) holds if g is right-continuous and upper semi-continuous.

[5] See also the discussion following the theorem.

3 An interpretation of regularity

McAfee and McMillan (1987) observed that in the smooth case, regularity is equivalent to the strict convexity of $1/(1-F(x))$. For a direct proof of this fact, assume that F is twice differentiable. Then

$$\frac{\partial J_f(x)}{\partial x} = \frac{\partial}{\partial x}\left(x - \frac{1-F(x)}{f(x)}\right) = 1 + \frac{f(x)^2 + (1-F(x))f'(x)}{f(x)^2}. \tag{5}$$

On the other hand,

$$\frac{\partial^2}{\partial x^2}\frac{1}{1-F(x)} = \frac{\partial}{\partial x}\frac{f(x)}{(1-F(x))^2} = \frac{(1-F(x))f'(x) + 2f(x)^2}{(1-F(x))^3}, \tag{6}$$

i.e., the respective signs of $J_f'(x)$ and $(1/(1-F(x)))''$ coincide.

It apparently went unnoticed that the above characterization implies the following statistical interpretation of regularity. Imagine a large number M of machines which fail one after another at rate $f(x)$. Pick one machine from the population, and assume it has been functional up to time x. By the law of large numbers, there are about $M(1-F(x))$ machines left. But all machines are ex-ante identical, so the uniform likelihood for the chosen machine to be *the next* to fail is $l(x) \approx 1/M(1-F(x))$. By the *zoom rate*, we mean the rate at which this likelihood grows over time (as a consequence of other machines failing). Regularity then requires the zoom rate to be increasing over time.[6]

To see the interpretation at work, recall that any regular type distribution remains regular after arbitrary truncations.[7] This is quite obvious for truncations from below because dropping a subpopulation consisting of all machines that stop working before some specified time x_0 obviously does not affect the later development of the zoom rate. For truncations from above, the intuition is as follows. Since the zoom rate is increasing, the population must shrink sufficiently fast to compensate for any slow-down in the development of failures. In this situation, dropping a subpopulation consisting of

[6] More formally, let $m \geq 1$ denote the exact number of machines that are still working at time x. Then, the likelihood for a given machine to be the next to fail is

$$\begin{aligned} l(x) &= \sum_{m=1}^{M} \frac{1}{m}\binom{M-1}{m-1}(1-F(x))^{m-1}F(x)^{M-m} \\ &= \frac{1}{M(1-F(x))}\sum_{m=1}^{M}\binom{M}{m}(1-F(x))^{m}F(x)^{M-m} \\ &= \frac{1-F(x)^M}{M(1-F(x))}. \end{aligned}$$

Therefore, for x kept fixed, $l(x)$ is indeed asymptotically equivalent to $1/M(1-F(x))$ as $M \to \infty$. Further, one can check that $\partial l/\partial x \approx f(x)/M(1-F(x))^2$. This follows from differentiating the precise expression for $l(x)$ derived above.

[7] Cf. Hafalir and Krishna (2008), or Virág (2011).

all machines that survive beyond some time x_1 accelerates the population effect, while there is no impact on the failures occurring before time x_1. Thus, the zoom rate will be increasing also in the truncated distribution.[8]

4 A generalization

So far, we assumed that the density function is differentiable. However, this may be restrictive, e.g., when the distribution is a mixture or the result of endogenous decisions. To incorporate such possibilities, smoothness will be replaced by a somewhat weaker assumption.

Consider a density $f \geq 0$ on some interval $X \subseteq \mathbb{R}$. Without loss of generality, we assume that f is strictly positive in the interior of X. Indeed, if $f(x) = 0$ at some interior point x, then $J_f(x) = -\infty$, and J_f cannot be increasing. We will say that f satisfies the *Cantor-Lebesgue condition* (CL) if f is right-continuous in the interior of X, upper semi-continuous, and satisfies $\overline{f}^{+} > -\infty$ except possibly at a countable set. This condition is quite weak. For example, it is satisfied for right-continuous, piecewise differentiable densities that do not possess downward jumps.[9]

The following auxiliary result can be seen as a generalization of the smooth characterization of regularity. Note, however, that it concerns weakly increasing virtual valuations.

Lemma 4.1 *Let $f > 0$ be a density on some interval $X \subseteq \mathbb{R}$, and assume that condition (CL) holds. Then, $J_f(x)$ is nondecreasing if and only if $1/(1 - F(x))$ is convex.*

Proof The right-hand upper Dini derivative of $J_f(x) = x - (1 - F(x))/f(x)$ is given by

$$\overline{J}_f^{+}(x) = 1 - \limsup_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left\{ \frac{1 - F(x+\varepsilon)}{f(x+\varepsilon)} - \frac{1 - F(x)}{f(x)} \right\} \tag{7}$$

$$= 1 + \limsup_{\varepsilon \to 0+} \frac{1}{f(x+\varepsilon)} \left\{ \frac{F(x+\varepsilon) - F(x)}{\varepsilon} + \frac{1 - F(x)}{f(x)} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \right\}. \tag{8}$$

Clearly, the derivative of F is a.e. well-defined with $F' = f$. Hence, noting that f is right-continuous,

$$\overline{J}_f^{+}(x) = 2 + \frac{(1 - F(x))\overline{f}^{+}(x)}{f(x)^2} \tag{9}$$

[8] The zoom rate formulation of increasing virtual valuations has been taken up already by Szech (2011) to predict over- and underinvestment in attracting bidders to an auction.

[9] Monteiro and Svaiter (2010) study optimal design for *arbitrary* distributions. For example, the support of the distribution may have gaps, and there may be mass points. Obviously, there is no role for regularity under such general conditions.

a.e. in X. Let $\vartheta_f(x) = f(x)/(1-F(x))^2$. Since F is continuous, the right-hand upper Dini derivative of $\vartheta_f(x)$ is given by

$$\overline{\vartheta}_f^+(x) = \limsup_{\varepsilon\to 0+} \frac{1}{\varepsilon}\left\{\frac{f(x+\varepsilon)}{(1-F(x+\varepsilon))^2} - \frac{f(x)}{(1-F(x))^2}\right\} \tag{10}$$

$$= \frac{1}{(1-F(x))^2} \limsup_{\varepsilon\to 0+}\left\{\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right.$$
$$\left. -\frac{f(x)}{(1-F(x))^2}\frac{(1-F(x+\varepsilon))^2-(1-F(x))^2}{\varepsilon}\right\}. \tag{11}$$

Hence,

$$\overline{\vartheta}_f^+(x) = \frac{\overline{f}^+(x)(1-F(x)) + 2f(x)^2}{(1-F(x))^3} \tag{12}$$

a.e. in X. Comparing (9) and (12) shows that $\overline{J}_f^+(x)$ and $\overline{\vartheta}_f^+(x)$ share the same sign a.e. in X.

"Only if." Assume that J_f is monotone. Then $\overline{J}_f^+ \geq 0$ on X, and therefore, $\overline{\vartheta}_f^+(x) \geq 0$ a.e. in X. An inspection of (11) shows that ϑ_f satisfies condition (CL). Hence, by Theorem 2.2, ϑ_f is nondecreasing. Thus, any integral of ϑ_f is convex, in particular $1/(1-F(x))$.

"If." Conversely, assume that $1/(1-F(x))$ is convex. Then the left derivative of $1/(1-F(x))$ is well-defined in the interior of X and monotone. But a.e. in X, the left derivative of $1/(1-F(x))$ is given by ϑ_f. Thus, $\overline{\vartheta}_f^+(x) \geq 0$ a.e. in X. As shown above, this implies $\overline{J}_f^+(x) \geq 0$ a.e. in X. One can check using (8) that J_f satisfies condition (CL). Therefore, by another application of Theorem 2.2, J_f is monotone increasing. □

5 A condition on the density

This section contains the key result of the paper. The Prékopa–Borell theorem is used to derive a tight criterion for regularity on the underlying density function. To deal with strict monotonicity, and to allow for modifications of the regularity assumption, we will write

$$J_f(x,a,b) = ax - \frac{b-F(x)}{f(x)}, \tag{13}$$

where $a, b \in \mathbb{R}$. Note that $J_f(x,1,1) \equiv J_f(x)$.

Theorem 5.1 *Let $f > 0$ be a density on some interval $X \subseteq \mathbb{R}$, and $a > -1$. Then $J_f(x,a,b)$ is weakly increasing in x [strictly increasing in x] for any $b \in [0,1]$ if f is ρ-concave [strongly ρ-concave] for $\rho = -\frac{a}{1+a}$. In particular, $J_f(x)$ is strictly increasing if f is strongly $(-\frac{1}{2})$-concave.*

Table 1 Distributions with strongly $(-\frac{1}{2})$-concave density function

Name of distribution	Interval X	P.d.f. $f(x)$	C.d.f. $F(x)$	Concavity ρ
Any with log-concave density	See Bagnoli and Bergström (2005, Table 1)			≥ 0
Pareto ($\beta > 1$)	$[1;\infty)$	$\beta x^{-\beta-1}$	$1 - x^{-\beta}$	$-\frac{1}{\beta+1}$
Log-normal[a] ($\sigma_L^2 < 2, \mu_L \in \mathbb{R}$)	$[0;\infty)$	$\propto \frac{1}{x}\exp\left(-\frac{(\ln x-\mu_L)^2}{2\sigma_L^2}\right)$	$*$	$-\frac{\sigma_L^2}{4}$
Student[a] ($n > 1$)	$\mathbb{R}$	$\propto (1+x^2/n)^{-\frac{n+1}{2}}$	$*$	$-\frac{1}{n+1}$
Cauchy[b]	$\mathbb{R}$	$\frac{1}{\pi(1+x^2)}$	$\frac{1}{2}+\frac{\arctan x}{\pi}$	$-\frac{1}{2}$
F distribution[a] ($m_1 \geq 2, m_2 > 2$)	$[0;\infty)$	$\propto \frac{x^{\frac{m_1}{2}-1}}{(m_1x+m_2)^{\frac{m_1+m_2}{2}}}$	$*$	$-\frac{2}{m_2+2}$
Mirror-image of Pareto ($\beta > 1$)	$(-\infty;-1]$	$\beta(-x)^{-\beta-1}$	$(-x)^{-\beta}$	$-\frac{1}{\beta+1}$
Log-logistic ($\beta > 1$)	$[0;\infty)$	$\frac{\beta x^{\beta-1}}{(1+x^\beta)^2}$	$\frac{x^\beta}{1+x^\beta}$	$-\frac{1}{\beta+1}$
Inverse gamma ($\alpha > 1$)	$[0;\infty)$	$\frac{\exp(-1/x)}{\Gamma(\alpha)x^{\alpha+1}}$	$*$	$-\frac{1}{\alpha+1}$
Inverse Chi-squared ($\upsilon > 2$)	$[0;\infty)$	$\frac{x^{-(\upsilon/2)-1}}{2^{\upsilon/2}\Gamma(\upsilon/2)}\exp\left(-\frac{1}{2x}\right)$	$*$	$-\frac{2}{\upsilon+2}$
Beta prime[a] ($\alpha \geq 1, \beta > 1$)	$[0;\infty)$	$\propto x^{\alpha-1}(1+x)^{-\alpha-\beta}$	$*$	$-\frac{1}{\beta+1}$
Pearson ($b_2 > -\frac{1}{2}$)	See the Appendix			b_2

[a] The symbol $\propto$ indicates that the density function, for fixed parameters, is proportional to the term given in the table; for cumulative distribution functions marked with $*$, there is no closed-form representation
[b] The density function of the Cauchy distribution is strongly $(-\frac{1}{2})$-concave on any compact interval

Proof Assume that f is $(-\frac{a}{1+a})$-concave for some $a > -1$. Consider the mirror image density $g(y) = f(-y)$. Obviously, also g is $(-\frac{a}{1+a})$-concave. By Theorem 2.1, the integral $G(y) = 1 - F(-y)$ is $(-a)$-concave, and so is $1 - F(x)$. Since f is continuous on X with finite right derivative in the interior, condition (CL) holds. Therefore, in straightforward extension of Lemma 4.1, $J_f(x, a, 1)$ is nondecreasing. Similarly, $J_f(x, a, 0)$ is nondecreasing since $F(x)$ is $(-a)$-concave. The unbracketed part of the theorem follows now from noting that $J_f(x, a, b)$ is linear in b. If f is even strongly $(-\frac{a}{1+a})$-concave for some $a > -1$ then, by the first part of the proof, $J_f(x, a', b)$ is weakly increasing in x for some $a' \in (-1, a)$. Hence, $J_f(x, a, b) = J_f(x, a', b) + (a - a')x$ is strictly increasing in x. □

Thus, strong $(-\frac{1}{2})$-concavity of the density is sufficient for regularity. Since any log-concave function is strongly $(-\frac{1}{2})$-concave, Theorem 5.1 clearly implies the conventional log-concavity criterion.[10]

On the other hand, density functions certainly may be strongly $(-\frac{1}{2})$-concave without being log-concave. For example, as Table 1 shows, this is the case for the log-normal, Pareto, log-logistic, Student, Cauchy, F, beta prime, mirror-image Pareto,

[10] Theorem 5.1 can be applied also if the density function has finitely many convex kinks and jump discontinuities. In such cases, one requires strong $(-\frac{1}{2})$-concavity of f in each smooth segment, and strong (-1)-concavity of F just left of critical points. For a proof, one constructs a strongly $(-\frac{1}{2})$-concave extension of the density right of the critical point. The details are omitted.

Table 2 Distributions without strongly $(-\frac{1}{2})$-concave density function

Name of distribution	Interval X	P.d.f. $f(x)$	C.d.f. $F(x)$	Values J'_f	Costs K'_f
Power $(c < 1)$	$[0;1]$	cx^{c-1}	x^c	$\ngtr 0$	> 0
Weibull $(c < 1)$	$[0;\infty)$	$cx^{c-1}\exp(-x^c)$	$1-\exp(-x^c)$	$\ngtr 0$	> 0
Gamma $(c < 1)$	$[0;\infty)$	$\frac{x^{c-1}\exp(-x)}{\Gamma(c)}$	$*$	$\ngtr 0$	> 0
Chi-squared $(c < 2)$	$[0;\infty)$	$\frac{x^{(c-2)/2}\exp(-x/2)}{2^{c/2}\Gamma(c/2)}$	$*$	$\ngtr 0$	> 0
Chi $(c < 1)$	$[0;\infty)$	$\frac{x^{c-1}\exp(-x^2/2)}{2^{(c-2)/2}\Gamma(c/2)}$	$*$	$\ngtr 0$	> 0
Beta[a] $(\upsilon < 1 \text{ or } \omega < 1)$	$[0;1]$	$\propto x^{\upsilon-1}(1-x)^{\omega-1}$	$*$	Mixed	Mixed
Arc-sine	$[0;1]$	$\frac{1}{\pi\sqrt{x(1-x)}}$	$\frac{2}{\pi}\arcsin(x)$	$\ngtr 0$	$\ngtr 0$
Pareto $(\beta < 1)$	$[1;\infty)$	$\beta x^{-\beta-1}$	$1-x^{-\beta}$	$\ngtr 0$	> 0
Log-normal[a] $(\sigma_L^2 > 2)$	$[0;\infty)$	$\propto \frac{1}{x}\exp\left(-\frac{(\ln x-\mu_L)^2}{2\sigma_L^2}\right)$	$*$	Mixed	> 0
Student[a] $(n < 1)$	$\mathbb{R}$	$\propto \left(1+\frac{x^2}{n}\right)^{-\frac{n+1}{2}}$	$*$	$\ngtr 0$	$\ngtr 0$
F distribution[a] $(m_1 < 2 \text{ or } m_2 \leq 2)$	$[0;\infty)$	$\propto \frac{x^{\frac{m_1}{2}-1}}{(m_1x+m_2)^{\frac{m_1+m_2}{2}}}$	$*$	Mixed	> 0
Mirror-image of Pareto $(\beta < 1)$	$(-\infty;-1]$	$\beta(-x)^{-\beta-1}$	$(-x)^{-\beta}$	> 0	$\ngtr 0$
Log-logistic $(\beta < 1)$	$[0;\infty)$	$\frac{\beta x^{\beta-1}}{(1+x^\beta)^2}$	$\frac{x^\beta}{1+x^\beta}$	$\ngtr 0$	> 0
Inverse gamma $(\alpha < 1)$	$[0;\infty)$	$\frac{\exp(-1/x)}{\Gamma(\alpha)x^{\alpha+1}}$	$*$	$\ngtr 0$	> 0
Inverse Chi-squared $(\upsilon < 2)$	$[0;\infty)$	$\frac{x^{-(\upsilon/2)-1}}{\Gamma(\upsilon/2)}\exp\left(-\frac{1}{2x}\right)$	$*$	$\ngtr 0$	> 0
Beta prime[a] $(\alpha, \beta < 1)$	$[0;\infty)$	$\propto x^{\alpha-1}(1+x)^{-\alpha-\beta}$	$*$	Mixed	> 0
Pearson $(b_2 < -\frac{1}{2})$	See the Appendix			Mixed	Mixed

[a] The symbol $\propto$ indicates that the density function, for fixed parameters, is proportional to the term given in the table; for cumulative distribution functions marked with $*$, there is no closed-form representation

inverse gamma, inverse Chi-squared, and Pearson distributions.[11] Thus, Theorem 5.1 finds new regular distributions and settles, in particular, the case of distributions with log-normal shape. Conversely, Table 2 lists various distributions that lack a strongly $(-\frac{1}{2})$-concave density function. We write $J'_f > 0$ if $J'_f(x) > 0$ holds for all $x \in X$ and for all strictly positive parameter values in the range indicated in the leftmost column. Similarly, we write $J'_f \ngtr 0$ if for all such parameter values, there is some $x \in X$ such that $J'_f(x) < 0$. The entry "Mixed" is used when neither $J'_f > 0$ nor $J'_f \ngtr 0$ holds. It can be seen that most examples in Table 2 are never regular, regardless of parameter values. Overall, this clearly illustrates the tightness of Theorem 5.1.

[11] Further details regarding Tables 1 and 2 can be found in the Appendix.

In fact, strong $(-\frac{1}{2})$-concavity is the tightest condition possible in terms of generalized concavity. For instance, the density function $f(x) = (1+x)^{-2}$ on $\mathbb{R}_+$ is $(-\frac{1}{2})$-concave, but not strictly so, and the corresponding distribution is irregular.

6 Applications

This section illustrates the use of Theorem 5.1 in specific settings. In all cases considered, the criterion reduces the regularity condition to a condition on the density function alone, and thereby adds clarity to the scope of the findings.

6.1 Standard design problems

The revenue-maximizing mechanism in Myerson (1981) is a second-price auction with reserve price provided the underlying type distribution is regular. While previous conditions on the density function required log-concavity, the conclusion continues to hold if the density is strongly $(-\frac{1}{2})$-concave. Relatedly, Riley and Samuelson (1981) show that the optimal reserve price in a broad class of auctions can be found by setting the virtual valuation equal to the seller's reservation value. The resulting equation has a unique solution provided the distribution of types is regular. The range of densities can be widened as before. For a somewhat richer example, recall that in Baron and Myerson (1982), optimal regulation of a monopoly discriminates between cost types provided that $\theta + (1-\alpha)\frac{F(\theta)}{f(\theta)}$ is increasing in θ, where $\alpha \in [0, 1]$ is the policy weight of monopoly. Here, it suffices to assume that f is ρ-concave for $\rho > -\frac{1}{2-\alpha}$, which is less restrictive than log-concavity.

6.2 Two-sided markets and auctions with resale

An optimal trading mechanism exists in Myerson and Satterthwaite (1983) if the buyer's virtual valuation $J_f(x)$ and the seller's *virtual cost* $K_f(x) \equiv J_f(x, 1, 0) = x + \frac{F(x)}{f(x)}$ are both increasing in x. The function K_f is increasing provided condition (CL) holds and F is strongly (-1)-concave. The commonly made condition on the density is log-concavity, but strong $(-\frac{1}{2})$-concavity is sufficient.[12] Similar conclusions can be drawn for auctions with resale where regularity conditions ensure that monopoly and monopsony prices are unique (Cheng and Tan 2010).

6.3 Distribution of bids

Guerre et al. (2000) show that bids in a first-price auction can be rationalized as a Bayesian equilibrium under the independent private value paradigm whenever $x + \frac{1}{I}\frac{G(x)}{g(x)} = \frac{J_g(x,I,0)}{I}$ increases in x, where I is the number of bidders, G is the distribution of bids, and g is the corresponding density function. While the common condition on

[12] In fact, for unimodal densities, strong $(-\frac{1}{2})$-concavity is required only on the increasing tail of the density, which also explains the many positive findings in the rightmost column of Table 2.

the density would be log-concavity, any strongly ρ-concave density with $\rho = -\frac{I}{1+I}$ can be rationalized.

6.4 Affiliated types

Chung and Ely (2007) consider affiliated valuations $x_1, \ldots, x_N$, and show that a generalized hazard rate condition implies a single-crossing condition for virtual valuations. This allows them to provide a foundation for dominant-strategy mechanisms. We note that with continuous affiliated types, assuming that $f(x_1, \ldots, x_N)$ is strongly $(-\frac{1}{2})$-concave would be much less restrictive than the generalized hazard rate condition, but still ensure the single-crossing condition for virtual valuations.[13]

6.5 Multidimensional types with externalities

An object is sold to one of N buyers. With externalities, buyer i's type is a vector (s_i^i, s_{-i}^i) whose entries specify the respective payoff to i in case some buyer j, not necessarily different from i, obtains the good. The distribution of buyer i's type follows some density $f_i(s_i^i, s_{-i}^i)$. Jehiel et al. (1999) show that the revenue-maximizing standard anonymous mechanism that always transfers the object is a second-price auction with entry fee if a modified regularity condition holds. Specifically, for

$$g(z) = \int\limits_{\mathbb{R}^{N-1}} f_i(z + \frac{1}{N-1} \sum_{j \neq i} s_j^i, s_{-i}^i) \, \mathrm{d}s_{-i}^i, \tag{14}$$

the virtual valuation J_g needs to be increasing. By Theorem 5.1, it suffices that g is strongly $(-\frac{1}{2})$-concave. But from (14), the change of variables in the argument of f_i is an affine transformation of $\mathbb{R}^N$, which leaves generalized concavity unaffected. Therefore, the second-price auction with entry fee is optimal if all f_i are strongly $(-\frac{1}{N+1})$-concave.

7 Conclusion

A formal re-examination of Myerson (1981) regularity assumption has led to the results in two dimensions. First, a new statistical interpretation of regularity has been proposed, which is both simple and similar to the one that is known for the hazard rate. Second, existing criteria for regularity due to Bagnoli and Bergström (2005) and An (1998) have been refined, using expanded concepts of concavity as well as minimal conditions for monotonicity. In sum, these results illuminate the scope of the regularity assumption, and substantially widen the range of specifications available for applied work.

[13] Chung and Ely (2007) assume discrete type distributions, but their results could probably be extended to continuous distributions.

Appendix: Parameterized distributions

This Appendix outlines the derivations underlying Tables 1 and 2. The main tool is the following smooth criterion for ρ-concavity.

Lemma A.1 *Let $f > 0$ be twice continuously differentiable on some interval $X \subseteq \mathbb{R}$, with a discrete set X_1 over which $f'(x) = 0$. Then, for finite ρ, the function f is ρ-concave if and only if $r_f(x) \equiv -(\ln f(x))''/(\ln f(x))'^2 \geq \rho$ for all $x \in X \setminus X_1$.*

Proof A straightforward calculation shows that $r_f(x) = 1 - f(x)f''(x)/f'(x)^2$. Hence, $r_f(x) \geq \rho$ if and only if $f(x)f''(x) \leq (1-\rho)f'(x)^2$, provided $f'(x) \neq 0$. By continuity, this proves the assertion. □

Lemma A.1 reduces the determination of the global concavity parameter to a straightforward minimization problem. More specifically, to find the tightest parameter ρ for which a given f is ρ-concave, one calculates the minimum (or infimum) of $r_f(x)$ on X. Table 1 shows the results for selected examples. A particular case is the Pearson distribution. Its density function solves the differential equation $f'(x) = f(x)(x - x^M)/\chi(x)$ for $x^M \in \mathbb{R}$ and $\chi(x) = b_0 + b_1 x + b_2 x^2$, where $b_0, b_1, b_2 \in \mathbb{R}$. We focus on distributions with unbounded support and such that $\chi(x^M) < 0$. Then, f is b_2-concave. Thus, both J_f and K_f are increasing provided that $b_2 > -\frac{1}{2}$.

Table 2 shows examples of distributions that do not allow a strongly $(-\frac{1}{2})$-concave density function. Unless noted otherwise, all parameter values are strictly positive. The only regular example is the mirror-image Pareto distribution. The entries of the form $J_f' \ngtr 0$, $K_f' \ngtr 0$, or "mixed" have been established by direct calculation at boundary values. In some cases, numerical calculations have been used. The entries with $K_f' > 0$ are typically straightforward, e.g., when the density function is everywhere declining. Some cases, however, need additional arguments. For example, both the log-normal distribution and the F distribution (Finner and Roters 1997) possess a log-normal distribution function, even though the corresponding density functions are not log-normal. In other cases (inverse gamma distribution and inverse Chi-squared), the density function is log-concave for low values and decreasing for high values, which again is sufficient for $K_f' > 0$.

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